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Scaling symmetries, conservation laws and action principles in one-dimensional gas dynamics

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Abstract
Scaling symmetries of the planar, one-dimensional gas dynamic equations with adiabatic index γ are used to obtain Lagrangian and Eulerian conservation laws associated with the symmetries. The known Eulerian symmetry operators for the scaling symmetries are converted to the Lagrangian form, in which the Eulerian spatial position of the fluid element is given in terms of the Lagrangian fluid labels. Conditions for a linear combination of the three scaling symmetries to be a divergence or variational symmetry of the action are established. The corresponding Lagrangian and Eulerian form of the conservation laws are determined by application of Noether's theorem. A nonlocal conservation law associated with the scaling symmetries is obtained by applying a nonlocal symmetry operator to the scaling symmetry-conserved vector. An action principle incorporating known conservation laws using Lagrangian constraints is developed. Noether's theorem for the constrained action principle gives the same formulas for the conserved vector as the classical Noether theorem, except that the Lie symmetry vector field now includes the effects of nonlocal potentials. Noether's theorem for the constrained action principle is used to obtain nonlocal conservation laws. The scaling symmetry conservation laws only apply for special forms of the entropy of the gas.

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1. Introduction

There is an extensive literature on the Lie symmetries and Hamiltonian structure of the ideal gas dynamic equations (e.g. Zakharov and Kuznetsov (1984), Salmon (1988), Ibragimov (1994), Nutku (1987), Olver and Nutku (1988), Morrison (1998), Holm *et al* (1998), Hydon (2005), Bridges and Reich (2005), Gibbon *et al* (2006) and Marsden and Ratiu (1994)) and the magnetohydrodynamic (MHD) equations (e.g. Morrison (1982), Holm and Kupershmidt (1983), Padhye and Morrison (1996a, 1996b), Padhye (1998), Kuznetsov and Ruban (2000), Fuchs (1991), Grundland and Lalague (1995), Webb *et al* (2005)). The Lie point symmetry algebra of the ideal, compressible gas dynamic and MHD equations have been obtained by Fuchs (1991) and classified by Grundland and Lalague (1995). The symmetries obtained by Fuchs (1991) pertain to the Eulerian form of the equations (see also Ibragimov (1994)).

Sjöberg and Mahomed (2004) obtained nonlocal symmetries and conservation laws for the planar, one-dimensional gas dynamic equations from the cover system, consisting of the original equations, supplemented by known conservation laws and their associated pseudo-potentials (see also Akhatov *et al* (1991), Ibragimov *et al* (1998), Kara and Mahomed (2000, 2002), Agafonov (1996), Anco and Bluman (2002), Bluman and Cheviakov (2005), Bluman *et al* (2006), Bluman (2008), Cheviakov (2008) for related approaches). The use of Noether's theorem to derive conservation laws requires that the differential equation system admits a variational formulation or action principle. Anco and Bluman (2002) developed a direct method of finding conservation laws of a system of partial differential equations that applies for equations with no variational principle. Olver and Nutku (1988) obtained higher order conservation laws and multi-Hamiltonian structures for the planar, one-dimensional gas dynamic equations for the case of an isentropic polytropic equation of state.

Webb and Zank (2007) investigated the Lie point symmetries, the fluid relabeling symmetries and the scaling symmetries of the three-dimensional MHD equations. They converted the Eulerian symmetries to Lagrange label space, in which the Eulerian position coordinate $\mathbf{x}$ is a function of the Lagrange fluid labels $\mathbf{x}_0$ and time t (i.e $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$). Each Eulerian Lie point symmetry of the Galilean group was mapped onto an infinite class of symmetries in Lagrange label space, associated with the fluid relabeling symmetries (see, e.g., Padhye and Morrison, 1996a, 1996b, Padhye 1998). The infinitesimal symmetry generators V^t, V^x, V^y, V^z are the same in both the Eulerian and Lagrangian symmetry operators, where the symmetry generator V^{x_0} for the fluid relabeling symmetry satisfies an auxiliary set of equations in Lagrange label space. The conditions for the scaling symmetries to be a divergence or variational symmetry of the action were derived, and used to obtain conservation laws using Noether's theorem. These laws only apply for special initial data for the gas entropy and magnetic field distribution and have a complicated form.

One aim of this paper is to derive the conservation laws for the scaling symmetries of planar, one-dimensional gas dynamics for an ideal gas, with adiabatic index γ. This involves converting the Eulerian symmetries to their Lagrangian form, and by determining the conditions for a linear combination of the scaling symmetries to be a divergence or variational symmetry of the action. Noether's theorem is used to obtain the conservation laws. Using the methods of Sjöberg and Mahomed (2004), we obtain a nonlocal conservation law by applying a nonlocal symmetry operator to the scaling symmetry conserved vector. An action principle incorporating known conservation laws as Lagrangian constraints is developed. Noether's theorem for the constrained action principle gives formulas for the conserved density and current of the same form as the classical Noether theorem, except that the symmetry generators now include the effects of nonlocal potentials. The variational principle is used to obtain nonlocal conservation laws. Our analysis shows the importance of the dependence of the entropy distribution on the Lagrangian mass coordinate for the different symmetries.

2. One-dimensional gas dynamics

The time-dependent, compressible, inviscid equations of Eulerian gas dynamics in one Cartesian space coordinate x may be written in the form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{2.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + A(p,\rho)\frac{\partial u}{\partial x} = 0, \tag{2.3}$$

where u is the fluid velocity, assumed to be directed along the x-axis. If the equation of state for the gas is written in the form $S = f(p,\rho)$, one finds that $A(p,\rho) = c^2\rho$, where $c^2 = \partial p/\partial\rho = -f_\rho/f_p$ is the square of the adiabatic sound speed for the gas. For the case of a gas with entropy $S = C_v \ln[(p/p_1)/(\rho/\rho_1)^\gamma]$ where $\gamma = C_p/C_v$ is the ratio of the specific heats at constant pressure and volume, $A(p,\rho) = \gamma p$. We investigate this case in detail in the present paper. An alternative formulation uses the internal energy density relation $\varepsilon = \varepsilon(\rho, S)$ as the equation of state for the gas, in which case $p = \rho\partial\varepsilon/\partial\rho - \varepsilon$ and $\rho T = \partial\varepsilon/\partial S$ define the pressure and temperature of the gas, and $c = (\partial p/\partial\rho)^{1/2}$ is the adiabatic gas sound speed. For an ideal gas with adiabatic index γ, $\varepsilon = p/(\gamma - 1)$. Equation (2.3) is also equivalent to the entropy advection equation:

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0. \tag{2.4}$$

Both (2.3) and (2.4) are equivalent to the co-moving energy equation for the gas.

2.1. Eulerian Lie point symmetries

The Eulerian Lie point symmetries for the 1D gas dynamic equations (2.1)–(2.3) are listed for example in Ibragimov (1994), vol 1, chapter 13, section 13.1.12 (see also Ovsjannikov (1962)). For arbitrary $A(p,\rho)$, the Lie point symmmetry algebra is spanned by the vector fields:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \tag{2.5}$$

The vector fields X_1 and X_2 correspond to the time and space translation symmetries, X_3 corresponds to a scaling symmetry and X_4 to the Galilean boost symmetry. The scaling symmetry X_3 corresponds to the symmetry operator $X_{11} + X_{12}$ of Webb and Zank (2007) for three-dimensional MHD.

Depending on the equation of state as specified by $A(p,\rho)$, the equations may admit further symmetries. In particular for an ideal, constant adiabatic index γ, gas:

$$A(p,\rho) = \gamma p, \quad \varepsilon = \frac{p}{\gamma - 1}, \qquad S = C_v \ln\left[\left(\frac{p}{p_1}\right)\left(\frac{\rho}{\rho_1}\right)^\gamma\right], \tag{2.6}$$

equations (2.1)–(2.3) admit two further scaling symmetries, listed below:

$$X_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + 2\rho\frac{\partial}{\partial \rho}, \qquad X_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}. \tag{2.7}$$

The Lagrangian map is discussed in more detail in section 2.2.

2.2. The Lagrangian map

In Lagrangian fluid dynamics, the Eulerian fluid particle position $\mathbf{x}$ is regarded as a solution of the differential equation system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u}(\mathbf{x}, t), \tag{2.8}$$

where the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ is regarded as a given function of $\mathbf{x}$ and t (cf Courant and Friedrichs (1976) and Broer and Kobussen (1974)). The solution of the dynamical system (2.8) for $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$ (the coordinates (x_0, y_0, z_0) are known as Lagrangian fluid labels) has the form: $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$. If $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ is 1–1 and invertible, then $\mathbf{x}_0 = X_0(\mathbf{x}, t)$, where $\mathbf{X}_0$ describes the inverse function. The fluid velocity is given by $\mathbf{u} = \partial\mathbf{X}(\mathbf{x}_0, t)/\partial t$.

For one-dimensional gas dynamics in one Cartesian space coordinate x:

$$\mathrm{d}x = \frac{\partial x}{\partial x_0}\,\mathrm{d}x_0 + \frac{\partial x}{\partial t}\,\mathrm{d}t = \frac{\partial x}{\partial x_0}\left(\frac{\partial x_0}{\partial x}\,\mathrm{d}x + \frac{\partial x_0}{\partial t}\,\mathrm{d}t\right) + \frac{\partial x}{\partial t}\,\mathrm{d}t. \tag{2.9}$$

Equating the coefficients of dx and dt in (2.9) gives the equations

$$\frac{\partial x}{\partial x_0}\frac{\partial x_0}{\partial x} = 1, \qquad \frac{\partial x}{\partial t} + \frac{\partial x}{\partial x_0}\frac{\partial x_0}{\partial t} = 0. \tag{2.10}$$

From equations (2.10), it follows that

$$\frac{\partial x_0}{\partial t} + u\frac{\partial x_0}{\partial x} = 0, \tag{2.11}$$

showing that the Lagrangian label x_0 is advected with the flow.

The mass continuity equation (2.1) can be written in the form

$$\rho(x, t)\,\mathrm{d}x = \rho_0\,\mathrm{d}x_0 \qquad \text{or} \qquad \rho J = \rho_0, \tag{2.12}$$

where $\rho_0 = \rho(x_0, 0)$ and $J = \partial x/\partial x_0$ is the Jacobian of the Lagrange map between the Eulerian position of the fluid element and its Lagrangian label x_0. The Lagrangian mass coordinate h given by

$$h = \int_{-\infty}^{x} \rho(x', t)\,\mathrm{d}x' = \int_{-\infty}^{x_0} \rho(x_0', 0)\,\mathrm{d}x_0' \tag{2.13}$$

may be used instead of x_0 as a Lagrangian fluid label. Both $h = h(x_0)$ and the entropy $S(x_0)$ are advected with the flow. Differentiating (2.13) with respect to h, keeping t constant, we find

$$\tau = 1/\rho = x_h \qquad \text{and} \qquad \rho = 1/x_h, \tag{2.14}$$

for the specific volume τ and density ρ in terms of $x\ (h, t)$.

Proposition 2.1. *Consider the action principle:*

$$A = \iint \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t \equiv \iint \mathcal{L}_0\,\mathrm{d}h\,\mathrm{d}t, \tag{2.15}$$

where the Lagrangian densities $\mathcal{L}$ and $\mathcal{L}_0$ are given by

$$\mathcal{L} = \tfrac{1}{2}\rho x_t^2 - \varepsilon(\rho, S), \qquad \mathcal{L}_0 = x_h\mathcal{L} \equiv \tfrac{1}{2}x_t^2 - F(x_h, h), \qquad F(x_h, h) = \frac{\varepsilon}{\rho}. \tag{2.16}$$

The condition that the action is stationary: $\delta A/\delta x = 0$ gives the Lagrangian x-momentum equation for 1D ideal gas dynamics in the form

$$\frac{\delta A}{\delta x} = \frac{\partial\mathcal{L}_0}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}_0}{\partial x_t}\right) - \frac{\partial}{\partial h}\left(\frac{\partial\mathcal{L}_0}{\partial x_h}\right) \equiv -(x_{tt} + p_h) = 0, \tag{2.17}$$

where

$$p = \rho\varepsilon_\rho - \varepsilon = -\frac{\partial F(x_h, h)}{\partial x_h} \tag{2.18}$$

is the gas pressure (see Broer and Kobussen (1974) and Courant and Friedrichs (1976) for further details). The Eulerian momentum equation follows by multiplying (2.17) by $\rho = 1/x_h$ and noting that $x_{tt} = \mathrm{d}u/\mathrm{d}t = u_t + uu_x$ and $\rho p_h = p_x$.

Comment 1

For the adiabatic equation of state (2.6), the Lagrangian momentum equation (2.17) may be written in the form of a nonlinear wave equation for $x\,(h,t)$ of the form

$$x_{tt} = N_1 x_h^{-\gamma-1} \exp[\bar{S}(h)][\gamma x_{hh} - x_h \bar{S}_h], \tag{2.19}$$

where

$$p = N_1 x_h^{-\gamma} \exp(\bar{S}), \qquad N_1 = p_1 \rho_1^{-\gamma}, \qquad \bar{S} = \frac{S}{C_v}. \tag{2.20}$$

Comment 2

Introduce a potential w for $x\,(h,t)$ and then

$$\mathcal{L}_0 = \frac{1}{2} w_{ht}^2 - F(w_{hh}, h) \qquad \text{where} \qquad x = w_h. \tag{2.21}$$

The condition for the action to be stationary reduces to

$$\frac{\delta A}{\delta w} = \frac{\partial^2}{\partial h^2}(w_{tt} + p) = 0 \qquad \text{so that} \qquad (w_{tt} + p) = 0, \tag{2.22}$$

for an appropriate choice of gauge. For a gas with adiabatic index γ, (2.22) reduces to

$$w_{tt} + N_1 w_{hh}^{-\gamma} \exp[\bar{S}(h)] = 0. \tag{2.23}$$

For $\gamma = -1$ and $\bar{S} =$ const. (2.23) is essentially Laplace's equation. This result is related to nonlocal symmetries of the 1D planar gas dynamic equations for the case $\gamma = -1$ obtained by Akhatov *et al* (1991). We use w in section 6 in the constrained variational principle for the cover system (5.17) of Sjöberg and Mahomed (2004).

3. Lagrangian scaling symmetries

In this section, we transform the general scaling symmetry

$$X_{(s)} = \alpha_3 X_3 + \alpha_5 X_5 + \alpha_6 X_6 \tag{3.1}$$

to its Lagrangian form. This results in constraints on the entropy distribution $S = S(h)$, which are used in section 4 to determine when the symmetries (3.1) are divergence symmetries of the action.

3.1. The Lagrangian form of the symmetry operator $X_{(s)}$

From (2.5) to (2.7) the Eulerian form of the symmetry operator (3.1) for the scaling symmetries may be written in the form

$$X_{(s)} = V^t \frac{\partial}{\partial t} + V^x \frac{\partial}{\partial x} + V^u \frac{\partial}{\partial u} + V^\rho \frac{\partial}{\partial \rho} + V^p \frac{\partial}{\partial p}, \tag{3.2}$$

where

$$\begin{aligned} &V^t = (\alpha_3 + \alpha_5)t, \qquad V^x = \alpha_3 x, \qquad V^u = -\alpha_5 u, \\ &V^\rho = (2\alpha_5 + \alpha_6)\rho, \qquad V^p = \alpha_6 p. \end{aligned} \tag{3.3}$$

Proposition 3.1.

The extended Lie symmetry operator (3.2) in Lagrangian coordinates (t, h, x) has the form

$$\begin{aligned} \tilde{X}_{(s)} = V^t \frac{\partial}{\partial t} + V^h \frac{\partial}{\partial h} + V^x \frac{\partial}{\partial x} + V^{x_t} \frac{\partial}{\partial x_t} + V^{x_h} \frac{\partial}{\partial x_h} + V^{x_{tt}} \frac{\partial}{\partial x_{tt}} \\ + V^{x_{ht}} \frac{\partial}{\partial x_{ht}} + V^{x_{hh}} \frac{\partial}{\partial x_{hh}} + \cdots. \end{aligned} \tag{3.4}$$

The detailed solutions for the coefficients in (3.4) depend on the value of the parameters

$$\delta_1 = -[\gamma\alpha_5 + (\gamma-1)\alpha_6/2], \qquad \delta_2 = \alpha_3 + 2\alpha_5 + \alpha_6, \qquad \delta_3 = -(2\alpha_5 + \alpha_6). \tag{3.5}$$

There are four cases to consider, depending on whether δ_1 and/or δ_2 are zero or non-zero, which are listed below.

Case (i) $\delta_1 \neq 0$, $\delta_2 \neq 0$.

In this case the symmetry generators have the form

$$\begin{aligned} &V^t = (\alpha_3 + \alpha_5)t, \qquad V^x = \alpha_3 x, \qquad V^h = \delta_2(h + d_1), \\ &V^{x_t} = -\alpha_5 x_t, \qquad V^{x_h} = \delta_3 x_h, \qquad V^{x_{tt}} = -(\alpha_3 + 2\alpha_5)x_{tt}, \\ &V^{x_{hh}} = (\delta_3 - \delta_2)x_{hh}, \qquad V^{x_{ht}} = (\delta_3 - \alpha_3 - \alpha_5)x_{ht}. \end{aligned} \tag{3.6}$$

The entropy $S = S(h)$ is constrained to have the form

$$S = C_v\left(-2\nu \ln|h + d_1| + d_2\right), \qquad \nu = -\frac{\delta_1}{\delta_2}, \tag{3.7}$$

where and d_1 and d_2 are arbitrary constants.

Case (ii) $\delta_1 = 0$, $\delta_2 \neq 0$.

The symmetry generators are the same as in (3.6) except that $\delta_1 = 0$ restricts the values of α_3, α_5 and α_6. The entropy $S = S_1 =$ const. corresponds to isentropic gas dynamics.

Case (iii) $\delta_1 \neq 0$, $\delta_2 = 0$.

The symmetry generators have the same form as (3.6) except that $V^h = k_1 =$ const. For $k_1 \neq 0$ the entropy S is constrained to have the form

$$S = 2C_v\frac{\delta_1 h}{k_1} + k_2. \tag{3.8}$$

Case (iv) $\delta_1 = 0$, $\delta_2 = 0$.

The symmetry generators are given by (3.6) except that $V^h = k_1 =$ const. For $k_1 = 0$, (i.e. $V^h = 0$), there is no constraint on $S(h)$. The case $k_1 \neq 0$ is the isentropic case for which $S =$ const.

Proof. The proof uses the Lie extension formulas for the transformation of derivatives, where $x = x(h, t)$ gives the Eulerian position of the fluid element terms of the Lagrangian mass coordinate h and time t. For the Lie infinitesimal transformations:

$$t' = t + \epsilon V^t, \qquad h' = h + \epsilon V^h, \qquad x' = x + \epsilon V^x, \tag{3.9}$$

derivative x_i transforms as

$$x'_{i'} = x_i + \epsilon V^{x_i}, \qquad V^{x_i} = D_i(V^x) - D_i(V^j)x_j, \tag{3.10}$$

where we use the notation $(x^1, x^2) = (t, h)$. Similarly,

$$x'_{i'j'} = x_{ij} + \epsilon V^{x_{ij}}, \qquad V^{x_{ij}} = D_j(V^{xi}) - D_j(V^k)x_{ki}. \tag{3.11}$$

These formulas, coupled with the known Eulerian form of the scaling symmetries in (3.3) may be used to derive (3.6)–(3.8). □

Comment

For the case $\delta_1 \neq 0$ and $\delta_2 \neq 0$, with the equation of state

$$p = N_1 x_h^{-\gamma}(h + d_1)^{-2\nu}, \qquad N_1 = p_1\rho_1^{-\gamma}, \qquad \nu = -\frac{\delta_1}{\delta_2}, \tag{3.12}$$

the Lagrangian wave equation (2.19) has the form

$$G \equiv x_{tt} - N_1 x_h^{-\gamma-1}(h + d_1)^{-2\nu}\left(\gamma x_{hh} + \frac{2\nu x_h}{h + d_1}\right) = 0. \tag{3.13}$$

4. Conservation laws in one-dimensional gas dynamics

We show that the general scaling symmetry $X_{(s)}$ in (3.1) is a variational or divergence symmetry of the action (2.15), if

$$\alpha_5 + \alpha_6 + 2\alpha_3 = 0. \tag{4.1}$$

Noether's theorem gives the corresponding conservation law.

By integrating the group trajectories:

$$\frac{\mathrm{d}t}{V^t} = \frac{\mathrm{d}x}{V^x} = \frac{\mathrm{d}u}{V^u} = \frac{\mathrm{d}\rho}{V^\rho} = \frac{\mathrm{d}p}{V^p} = \mathrm{d}\epsilon, \tag{4.2}$$

we obtain the finite equations of the group in the form

$$\begin{aligned} &x' = x\exp(\alpha_3\epsilon), \qquad t' = t\exp[\epsilon(\alpha_3+\alpha_5)], \qquad u' = u\exp(-\alpha_5\epsilon),\\ &\rho' = \rho\exp[(2\alpha_5+\alpha_6)\epsilon], \qquad p' = p\exp(\alpha_6\epsilon). \end{aligned} \tag{4.3}$$

The finite transformations (4.3) leave the gas dynamic equations (2.1)–(2.3) invariant for the case $A(p,\rho) = \gamma p$. In fact

$$\begin{aligned} &\frac{\partial\rho'}{\partial t'} + \frac{\partial}{\partial x'}(\rho' u') = \exp[\epsilon(\alpha_5+\alpha_6-\alpha_3)]\left(\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u)\right),\\ &\rho'\frac{\mathrm{d}u'}{\mathrm{d}t'} + \frac{\partial p'}{\partial x'} = \exp[\epsilon(\alpha_6-\alpha_3)]\left(\rho\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial p}{\partial x}\right),\\ &\frac{\mathrm{d}p'}{\mathrm{d}t'} + \gamma p'\frac{\partial u'}{\partial x'} = \exp[\epsilon(\alpha_6-\alpha_3-\alpha_5)]\left(\frac{\mathrm{d}p}{\mathrm{d}t} + \gamma p\frac{\partial u}{\partial x}\right), \end{aligned} \tag{4.4}$$

where $\mathrm{d}/\mathrm{d}t = \partial_t + u\partial_x$ is the Lagrangian time derivative. Thus, the transformed equations are satisfied, if the original equations (2.1)–(2.3) are satisfied.

The dimensions of the action (2.15) transform as

$$[A'] = [p'x't'] = \exp\{(\alpha_5+\alpha_6+2\alpha_3)\epsilon\}[pxt], \tag{4.5}$$

which implies that for the action to be invariant under scaling transformations (4.3) requires $\alpha_5 + \alpha_6 + 2\alpha_3 = 0$. Thus, condition (4.1) follows from dimensional analysis of the action. A more formal derivation of (4.1) follows from Noether's theorem (see below).

4.1. Noether's theorem

A version of Noether's first theorem, sufficiently general for our purposes, is given below.

Noether's first theorem

If the action

$$A = \iint \mathcal{L}_0(t, h, x, x_t, x_h)\,\mathrm{d}h\,\mathrm{d}t \tag{4.6}$$

is invariant under the infinitesimal Lie transformation:

$$t' = t + \epsilon V^t, \qquad h' = h + \epsilon V^h, \qquad x' = x + \epsilon V^x, \tag{4.7}$$

and the divergence transformations:

$$\mathcal{L}_0' = \mathcal{L}_0 + \epsilon(D_t\Lambda^t + D_h\Lambda^h), \tag{4.8}$$

to $O(\epsilon^2)$, then the differential equation

$$E_x(\mathcal{L}_0) = \frac{\delta A}{\delta x} = \frac{\partial\mathcal{L}_0}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}_0}{\partial x_t}\right) - \frac{\partial}{\partial h}\left(\frac{\partial\mathcal{L}_0}{\partial x_h}\right) = 0 \tag{4.9}$$

admits the conservation law

$$D_t(I_0) + D_h(I_1) = 0, \tag{4.10}$$

where

$$I_0 = W^t + \mathcal{L}_0 V^t + \Lambda^t, \qquad I_1 = W^h + \mathcal{L}_0 V^h + \Lambda^h \tag{4.11}$$

are the conserved density I_0 and flux I_1, and

$$\hat{V}^x = V^x - (V^t D_t + V^h D_h)x \tag{4.12}$$

is the canonical Lie symmetry generator (i.e. the Lie transformation $x' = x + \epsilon \hat{V}^x$, $t' = t$ and $h' = h$ that is equivalent to (4.7)). The quantities

$$W^t = \hat{V}^x \frac{\partial \mathcal{L}_0}{\partial x_t}, \qquad W^h = \hat{V}^x \frac{\partial \mathcal{L}_0}{\partial x_h}, \tag{4.13}$$

are the surface vector terms that arise under the canonical Lie transformation, for which the variation of A is given by

$$\delta A = \int\int (\hat{V}^x E_x(\mathcal{L}_0) + D_t W^t + D_h W^h)\, \mathrm{d}h\, \mathrm{d}t. \tag{4.14}$$

The condition that the Lie transformations (4.7) and the divergence transformation (4.8) leave the action (4.6) invariant to $O(\epsilon)$ is

$$\tilde{X}\mathcal{L}_0 + \mathcal{L}_0[D_t V^t + D_h V^h] + D_t \Lambda^t + D_h \Lambda^h = 0, \tag{4.15}$$

where $\tilde{X}$ is the extended Lie symmetry operator corresponding to the symmetries (4.7). If condition (4.15) is satisfied, then (4.9) admits the conservation law (4.10) (see, e.g., Ibragimov (1985), Bluman and Kumei (1989) and Olver (1993) for details).

4.2. Conservation laws

Proposition 4.1. *For the gas dynamic action principle (2.15) for a gas with the equation of state (2.6):*

$$\tilde{X}\mathcal{L}_0 + (D_t V^t + D_h V^h)\mathcal{L}_0 = (\alpha_5 + \alpha_6 + 2\alpha_3)\mathcal{L}_0. \tag{4.16}$$

Thus if

$$\alpha_5 + \alpha_6 + 2\alpha_3 = 0, \tag{4.17}$$

the symmetry operator X is a variational symmetry of the action (2.15), in which case the 1D gas dynamic system admits the conservation law (4.10), where the conserved vector $\mathbf{T} = (I_0, I_1)$ *is given by*

$$I_0 = \hat{V}^x u + V^t \mathcal{L}_0, \qquad I_1 = \hat{V}^x p + V^h \mathcal{L}_0. \tag{4.18}$$

The detailed form of $\mathbf{T}$ *depends on whether* δ_1 *and* δ_2 *are zero or non-zero.*

Proof. The proof follows immediately from the identities

$$\tilde{X}\mathcal{L}_0 = -2\alpha_5\mathcal{L}_0, \qquad [D_t V^t + D_h V^h]\mathcal{L}_0 = (2\alpha_3 + 3\alpha_5 + \alpha_6)\mathcal{L}_0. \tag{4.19}$$

The detailed form of $\hat{V}^x$, V^t, V^h for the different cases for δ_1 and δ_2 in proposition (4.1), in order that a conservation law of the form (4.10) is obtained are listed below. □

Case (i) $\delta_1 \neq 0$, $\delta_2 \neq 0$.

Taking into account (4.16) for a conservation law to apply, we obtain

$$\hat{V}^x = \alpha_3 x - (\alpha_3+\alpha_5)tx_t - (\alpha_5-\alpha_3)(h+d_1)x_h,$$
$$V^t = (\alpha_3+\alpha_5)t, \qquad V^h = (\alpha_5-\alpha_3)(h+d_1). \tag{4.20}$$

The Lagrangian density $\mathcal{L}_0$ and gas pressure p are given by

$$\mathcal{L}_0 = \frac{1}{2}u^2 - \frac{p}{(\gamma-1)\rho} \equiv \frac{1}{2}x_t^2 - \frac{N_1 x_h^{1-\gamma}}{\gamma-1}(h+d_1)^{-2\nu},$$
$$p = N_1 x_h^{-\gamma}(h+d_1)^{-2\nu}, \qquad \nu = -\frac{\delta_1}{\delta_2}. \tag{4.21}$$

Note that ν depends on the ratio α_3/α_5. The conserved density I_0 and flux I_1 are

$$I_0 = \{\alpha_3 x - (\alpha_3+\alpha_5)tx_t - (\alpha_5-\alpha_3)(h+d_1)x_h]\}\, x_t + (\alpha_3+\alpha_5)t\mathcal{L}_0,$$
$$I_1 = \{\alpha_3 x - (\alpha_3+\alpha_5)tx_t - (\alpha_5-\alpha_3)(h+d_1)x_h]\}\, p + (\alpha_5-\alpha_3)(h+d_1)\mathcal{L}_0. \tag{4.22}$$

Case (ii) $\delta_1 = 0, \delta_2 \neq 0$.

In this case the conditions $\delta_1 = 0$ and $\alpha_5+\alpha_6+2\alpha_3 = 0$ imply

$$(\alpha_3,\alpha_5,\alpha_6) = \alpha_6\left(-\frac{\gamma+1}{4\gamma}, -\frac{(\gamma-1)}{2\gamma}, 1\right),$$
$$\hat{V}^x = \alpha\left[\frac{\gamma+1}{3\gamma-1}x - \left(tx_t + \frac{\gamma-3}{3\gamma-1}(h+d_1)x_h\right)\right],$$
$$V^t = \alpha t, \qquad V^h = \frac{\alpha(\gamma-3)}{3\gamma-1}(h+d_1), \qquad \alpha = \alpha_3+\alpha_5,$$
$$\mathcal{L}_0 = \frac{1}{2}x_t^2 - \frac{N_1 x_h^{1-\gamma}\exp(\bar{S}_1)}{\gamma-1}, \qquad p = N_1 x_h^{-\gamma}\exp(\bar{S}_1), \tag{4.23}$$

in the conservation law (4.10). The gas is isentropic gas with $\bar{S} = \bar{S}_1 =$ const., $\delta_2 \neq 0$ and $\gamma \neq 3$. I_0 and I_1 (with $\alpha = 1$) are

$$I_0 = \left[\frac{\gamma+1}{3\gamma-1}x - \left(tx_t + \frac{\gamma-3}{3\gamma-1}(h+d_1)x_h\right)\right]x_t + t\mathcal{L}_0,$$
$$I_1 = \left[\frac{\gamma+1}{3\gamma-1}x - \left(tx_t + \frac{\gamma-3}{3\gamma-1}(h+d_1)x_h\right)\right]p + \frac{(\gamma-3)}{3\gamma-1}(h+d_1)\mathcal{L}_0. \tag{4.24}$$

Case (iii) $\delta_1 \neq 0, \delta_2 = 0$.

$$(\alpha_3,\alpha_5,\alpha_6) = \alpha\left(\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right),$$
$$\hat{V}^x = \alpha\left(\frac{1}{2}x - tx_t\right) - k_1 x_h, \qquad V^t = \alpha t, \qquad V^h = k_1 \neq 0,$$
$$\mathcal{L}_0 = \frac{1}{2}x_t^2 - \frac{N_1 x_h^{1-\gamma}}{\gamma-1}\exp\left(\frac{\alpha(\gamma-3)h}{2k_1}\right), \qquad p = N_1 x_h^{-\gamma}\exp\left(\frac{\alpha(\gamma-3)h}{2k_1}\right), \tag{4.25}$$
$$I_0 = \left[\alpha\left(\frac{1}{2}x - tx_t\right) - k_1 x_h\right]x_t + \alpha t\mathcal{L}_0,$$
$$I_1 = \left[\alpha\left(\frac{1}{2}x - tx_t\right) - k_1 x_h\right]p + k_1\mathcal{L}_0.$$

Case (iv) $\delta_1 = 0$, $\delta_2 = 0$.

$$\begin{aligned}
&\gamma = 3, \qquad (\alpha_3, \alpha_5, \alpha_6) = \alpha\left(\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right),\\
&\hat{V}^x = \alpha\left(\frac{1}{2}x - tx_t\right), \qquad V^t = \alpha t, \qquad V^h = 0,\\
&\mathcal{L}_0 = \frac{1}{2}x_t^2 - \frac{N_1 x_h^{-2}}{2}\exp(\bar{S}(h)), \qquad p = N_1 x_h^{-3}\exp(\bar{S}(h)),\\
&I_0 = \alpha\left[\left(\frac{1}{2}x - tx_t\right)x_t + t\mathcal{L}_0\right], \qquad I_1 = \alpha\left(\frac{1}{2}x - tx_t\right)p.
\end{aligned} \tag{4.26}$$

Proposition 4.2. *The Lagrangian conservation law (4.10) can be written in the Eulerian form:*

$$\frac{\partial F_0}{\partial t} + \frac{\partial F_1}{\partial x} = 0, \tag{4.27}$$

where the conserved vector (F_0, F_1) *is given by*

$$F_0 = \rho I_0, \qquad F_1 = \rho u I_0 + I_1. \tag{4.28}$$

Proof. Use the Eulerian mass continuity equation (2.1) and use $\rho = \partial h/\partial x = 1/x_h$. □

Example. For case (i) with $\delta_1 \neq 0$, $\delta_2 \neq 0$, the conserved vector (F_0, F_1) is given by

$$\begin{aligned}
F_0 &= \rho u\left[\alpha_3 x - (\alpha_3+\alpha_5)tu + \frac{\alpha_5-\alpha_3}{\rho}\exp\left(-\frac{\bar{S}}{2\nu}\right)\right] + (\alpha_3+\alpha_5)t\mathcal{L},\\
F_1 &= (p+\rho u^2)\left[\alpha_3 x - (\alpha_3+\alpha_5)tu + \frac{(\alpha_5-\alpha_3)}{\rho}\exp\left(-\frac{\bar{S}}{2\nu}\right)\right]\\
&\quad + \mathcal{L}\left[(\alpha_3+\alpha_5)tu + \frac{(\alpha_5-\alpha_3)}{\rho}\exp\left(-\frac{\bar{S}}{2\nu}\right)\right],
\end{aligned} \tag{4.29}$$

where $\mathcal{L} = \rho\mathcal{L}_0$ is the Eulerian Lagrangian density of the fluid.

5. Nonlocal symmetries and conservation laws

In this section, we use the scaling symmetry conservation law (4.22) to derive a nonlocal conservation law based on the work of Sjöberg and Mahomed (2004) and Kara and Mahomed (2002).

Kara and Mahomed (2002) and Sjöberg and Mahomed (2004) studied the effects of Lie symmetry operators, *X*, of a partial differential equation system:

$$F^\beta(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(m)}) = 0, \qquad 1 \leqslant \beta \leqslant p, \tag{5.1}$$

on a family of conserved vectors $\{\mathbf{T}_\gamma\}$ for the differential equation system, satisfying the conservation laws

$$\nabla\cdot\mathbf{T}_\gamma = D_i T^i_\gamma = 0, \qquad 1 \leqslant \gamma \leqslant r. \tag{5.2}$$

Here we use the Einstein summation convention and i labels the independent variables x^i, $(1 \leqslant i \leqslant n)$, D_i is the total partial differential operator with respect to x^i and γ labels the family of conservation laws. They showed that (5.1) also has conserved vectors $\mathbf{T}^*_\gamma$ defined by the formula

$$T^{*i}_\gamma = \tilde{X}\left(T^i_\gamma\right) + T^i_\gamma D_j\xi^j - T^j D_j\xi^i, \tag{5.3}$$

where $\tilde{X}$ is the extended Lie symmetry operator corresponding to X and the $\{\xi^i : 1 \leqslant i \leqslant n\}$ are the infinitesimal generators for changes of the x^i (i.e. $x'^i = x^i + \epsilon\xi^i$, $u'^\alpha = u^\alpha + \epsilon V^{u^\alpha}$ are the infinitesimal Lie transformations). The result (5.3) is derived in appendix , using the algebra of exterior differential forms.

For the case of two independent variables (x^1, x^2) (i.e. $n = 2$), one can introduce potentials $\tilde{u}^\gamma$ associated with each conservation law, such that

$$T^1_\gamma = D_2\tilde{u}^\gamma, \qquad T^2_\gamma = -D_1\tilde{u}^\gamma. \tag{5.4}$$

Then the conservation law is a consequence of the equality of mixed partial derivatives of the $\tilde{u}^\gamma$. It turns out that the symmetry operator X associated with the conservation law $D_i T^{*i}_\gamma = 0$ has a Lie extension operator of the form

$$\tilde{X} = X + u^{*\gamma}\frac{\partial}{\partial\tilde{u}^\gamma}, \tag{5.5}$$

where $u^{*\gamma}$ is the potential corresponding to T^*_γ, i.e.

$$\mathbf{T}^*_\gamma = (D_2 u^{*\gamma}, -D_1 u^{*\gamma}). \tag{5.6}$$

Further details of this theory can be found in the above papers.

The main point here is that known conserved vectors $\mathbf{T}_\gamma$ coupled with a Lie symmetry operator X of (5.1) can be used to generate other conservation laws of (5.1) via the transformation (5.3). The result (5.3) is well known for canonical Lie symmetry operators $\hat{X} = \tilde{X} - \xi^i D_i$ (e.g. Ibragimov (1985)) which correspond to Lie transformations of the form:

$$x'^i = x^i, \qquad u'^\alpha = u^\alpha + \epsilon\hat{V}^{u^\alpha}, \qquad \hat{V}^{u^\alpha} = V^{u^\alpha} - \xi^i u^\alpha_i. \tag{5.7}$$

The canonical transformations (5.7) are equivalent to the transformations $x'^i = x^i + \epsilon\xi^i$, $u'^\alpha = u^\alpha + \epsilon V^{u^\alpha}$. The extended Lie canonical symmetry operators corresponding to (5.7) commute with the D_i (i.e. $[\hat{X}, D_i] = 0$). It then follows that if T^i is a conserved vector, then $T^{*i} = \hat{X}T^i$ is also a conserved vector (note $\hat{X}(D_i T^i) = D_i(\hat{X}T^i) = D_i T^{*i}$).

Sjöberg and Mahomed (2004) applied these ideas to the conservation laws of one-dimensional planar gas dynamics, using three standard forms of the equations: (a) the Eulerian form (2.1)–(2.3), (b) an intermediate form in which x and t are the independent variables, and the Lagrangian mass coordinate h, fluid velocity u and gas pressure p are the dependent variables, and (c) the Lagrangian form of the equations in which h and t are the independent variables, and $q = 1/\rho$, u and p are the dependent variables (see also Akhatov *et al* (1991)).

The Lagrangian form of the equations can be written in the form:

$$q_t - u_h = 0, \tag{5.8}$$

$$u_t + p_h = 0, \tag{5.9}$$

$$p_t + \frac{\gamma p}{q}u_h = 0, \tag{5.10}$$

where

$$q = x_h = \frac{1}{\rho}. \tag{5.11}$$

Equation (5.10) is equivalent to the entropy advection equation for the case of a gas with adiabatic index γ and with the equation of state

$$p = p_0(q/q_0)^{-\gamma}\exp[\bar{S}(h)], \tag{5.12}$$

where $\bar{S} = S/C_v$ is a normalized version of the gas entropy. For the scaling symmetry conservation law (4.22), the equation of state from (4.21) has the form

$$p = N_1 q^{-\gamma} h^{-2\nu}, \qquad \nu = \frac{[(\gamma+1)/2]\alpha_5 - (\gamma-1)\alpha_3}{\alpha_5 - \alpha_3}, \tag{5.13}$$

where we have made the replacement $h + d_1 \to h$ in the formulas of section 4. The dependence of the parameter ν on α_3 and α_5 in (5.13) is a consequence of the requirement that $\alpha_5 + \alpha_6 + 2\alpha_3 = 0$ for the scaling symmetries to give a conservation law.

Sjöberg and Mahomed (2004) consider the basic gas dynamic equations (5.8)–(5.10) supplemented by known conservation laws, of the form

$$\nabla\cdot\mathbf{T}_\gamma \equiv \frac{\partial T^1_\gamma}{\partial t} + \frac{\partial T^2_\gamma}{\partial h} = 0, \qquad \mathbf{T}_\gamma = \left(\tilde{u}^\gamma_h, -\tilde{u}^\gamma_t\right), \tag{5.14}$$

with similar conservation laws for the Eulerian and intermediate systems. The total system of equations consisting of the original equations (5.8)–(5.10) and a class of conservation laws is known as the cover system. The conservation laws for the system can be written in the potential form:

$$\tilde{u}^\gamma_h - T^1_\gamma = 0, \qquad \tilde{u}^\gamma_t + T^2_\gamma = 0, \tag{5.15}$$

where the $\tilde{u}^\gamma$ are the potentials (pseudo-potentials) associated with the conservation laws, and T^1_γ and T^2_γ are the conserved density and flux, respectively.

For the mass, energy, momentum and center of mass conservation laws, the conserved vectors $\mathbf{T}_\gamma$ are

$$\begin{aligned} &\mathbf{T}_1 = (q, -u), \qquad \mathbf{T}_2 = \left(\frac{1}{2}u^2 + \frac{pq}{\gamma-1}, pu\right),\\ &\mathbf{T}_3 = (u, p), \qquad \mathbf{T}_4 = \left(tu - \tilde{u}^1, tp\right), \end{aligned} \tag{5.16}$$

respectively (note $\tilde{u}^1 \equiv x(h, t)$). In this case, the cover system consists of the equations

$$\begin{aligned} &q_t - u_h = 0, \qquad u_t + p_h = 0, \qquad p_t + \frac{\gamma p}{q} u_h = 0,\\ &\tilde{u}^1_h - q = 0, \qquad \tilde{u}^1_t - u = 0,\\ &\tilde{u}^2_h - \left(\frac{1}{2}u^2 + \frac{pq}{\gamma-1}\right) = 0, \qquad \tilde{u}^2_t + pu = 0,\\ &\tilde{u}^3_h - u = 0, \qquad \tilde{u}^3_t + p = 0,\\ &\tilde{u}^4_h - (tu - \tilde{u}^1) = 0, \qquad \tilde{u}^4_t + tp = 0. \end{aligned} \tag{5.17}$$

It turns out that the Lie point symmetries of the cover system (5.17) is a larger class than that of the original system (5.8)–(5.10). The symmetries consist of the usual Lie point symmetries $\{X_i : 1 \leqslant i \leqslant 6\}$ listed in (2.5)–(2.7) consisting of the Galilean symmetries $\{X_1, X_2, X_4\}$ and the scaling symmetries $\{X_3, X_5, X_6\}$ but converted to their Lagrangian form, plus the fluid relabeling symmetry $X_7 = \partial/\partial h$ (Sjöberg and Mahomed use a different labeling system than that used in the present paper). The detailed form of these symmetry operators for the Lagrangian cover system (5.17), including their action on the potentials $\tilde{u}^\gamma$, $(1 \leqslant \gamma \leqslant 4)$ are listed in Sjöberg and Mahomed (2004). However, Sjöberg and Mahomed also give a further symmetry operator

$$\begin{aligned} \tilde{X}_{11} = &-h^2\frac{\partial}{\partial h} - hp\frac{\partial}{\partial p} + (hu - \tilde{u}^3)\frac{\partial}{\partial u} + 3hq\frac{\partial}{\partial q} + (\tilde{u}^4 - \tilde{u}^3 t + \tilde{u}^1 h)\frac{\partial}{\partial \tilde{u}^1}\\ &- \frac{1}{2}(\tilde{u}^3)^2\frac{\partial}{\partial \tilde{u}^2} - \tilde{u}^3 h\frac{\partial}{\partial \tilde{u}^3} - \tilde{u}^4 h\frac{\partial}{\partial \tilde{u}^4}, \end{aligned} \tag{5.18}$$

which is a Lie point symmetry of the cover system (5.17), but is a nonlocal symmetry of the original system (5.8)–(5.10). A list of the extended symmetry operators $\{\tilde{X}_i : i = 1(1)7,\ 11\}$ used in the present paper and their commutators is given in appendix .

We now ask if the nonlocal symmetry (5.18) is compatible with the equation of state (5.13) for $p = p(q, h)$ obtained for the scaling symmetry conservation laws? To this end, we note from (5.13) that the Lie transformations $p' = p + \epsilon V^p$, $h' = h + \epsilon V^h$ and $q' = q + \epsilon V^q$ compatible with (5.13) satisfy the equation

$$V^p = p\left(-\frac{\gamma}{q}V^q - \frac{2\nu}{h}V^h\right). \tag{5.19}$$

Using $V^q = 3hq$ and $V^h = -h^2$ from (5.18) in (5.19) gives

$$V^p = ph(2\nu - 3\gamma) = -ph \qquad \text{if} \qquad 2\nu = 3\gamma - 1. \tag{5.20}$$

Using the expression (5.13) for ν in (5.20) we find that the symmetry X_{11} in (5.18) is compatible with the scaling symmetry equation of state (5.13) if

$$\alpha_5 = \frac{(\gamma+1)\alpha_3}{2(\gamma-1)} \qquad \text{and} \qquad 2\nu = 3\gamma - 1. \tag{5.21}$$

With the choice of parameters (5.21), the scaling symmetry equation of state is compatible with the symmetry X_{11}. It is apparent from (5.12) that the entropy $\bar{S}(h) = -(3\gamma - 1)\ln(h) + \text{const.}$ in order that X_{11} is a symmetry in the general case.

Proposition 5.1. *The one-dimensional planar gas dynamic equations (5.8)–(5.11) for an equation of state*

$$p = N_1 q^{-\gamma} h^{-(3\gamma-1)}, \tag{5.22}$$

possesses a nonlocal conservation law of the form (5.14) with conserved density T^{*1} and flux T^{*2} given by

$$\begin{aligned} T^{*1} &= \alpha_3 u\tilde{u}^4 + \tilde{u}^3[-\alpha_3\tilde{u}^1 + (\alpha_5 - \alpha_3)hq + \alpha_5 tu] - (\alpha_5 - \alpha_3)h^2 qu, \\ T^{*2} &= \alpha_3 p\tilde{u}^4 + \tilde{u}^3\,[\alpha_5 tp - (\alpha_5 - \alpha_3)hu] + (\alpha_5 - \alpha_3)h^2\left(\frac{1}{2}u^2 - \frac{\gamma pq}{\gamma - 1}\right), \end{aligned} \tag{5.23}$$

where $\alpha_5 = (\gamma + 1)\alpha_3/[2(\gamma - 1)]$ as in (5.21), and the nonlocal potentials $\tilde{u}^1$, $\tilde{u}^3$ and $\tilde{u}^4$ satisfy the cover system (5.17).

Proof. The components of the scaling symmetry conserved vector $T_s = (I_0, I_1)$ in (4.22) may be expressed in the form

$$\begin{aligned} I_0 &= \alpha_3 xu - (\alpha_5 - \alpha_3)hqu - (\alpha_3 + \alpha_5)t\left[\frac{1}{2}u^2 + \frac{pq}{\gamma - 1}\right], \\ I_1 &= \alpha_3 xp - (\alpha_3 + \alpha_5)put + (\alpha_5 - \alpha_3)h\left[\frac{1}{2}u^2 - \frac{\gamma pq}{\gamma - 1}\right]. \end{aligned} \tag{5.24}$$

Using the nonlocal symmetry operator $\tilde{X}_{11}$ from (5.18) and the scaling symmetry conserved vector $T_s = (I_0, I_1)$ from (5.24) in (5.3) gives formulas (5.23) for (T^{*1}, T^{*2}). The parameters α_3 and α_5 must be chosen to satisfy (5.21) for X_{11} to be a symmetry of the cover system (5.17) with equation of state (5.22) for the gas. One can verify the conservation law (5.23) by straight forward differentiation to obtain the equation

$$\frac{\partial T^{*1}}{\partial t} + \frac{\partial T^{*2}}{\partial h} = \frac{(\alpha_5 - \alpha_3)pqh}{\gamma - 1}[2\nu - (3\gamma - 1)]. \tag{5.25}$$

For the parameter constraints (5.21), $2\nu = (3\gamma - 1)$, (5.25) is a conservation law. This completes the proof. □

6. A constrained variational principle

In this section, we obtain a constrained variational principle to describe the cover system (5.17) used by Sjöberg and Mahomed (2004) to investigate nonlocal symmetries of the planar, one-dimensional Lagrangian gas dynamic equations. There is a connection between the potentials $\tilde{u}^1$, $\tilde{u}^3$ and $\tilde{u}^4$ in the cover system (5.17) and the potential w introduced in (2.21) *et seq.*, namely

$$\tilde{u}^1 = w_h, \qquad \tilde{u}^3 = w_t, \qquad \tilde{u}^4 = t w_t - w. \tag{6.1}$$

Formulas (6.1) for $\tilde{u}^1$, $\tilde{u}^3$ and $\tilde{u}^4$ satisfy the cover system (5.17). However, it is not possible to express the potential $\tilde{u}^2$ in terms of w. This suggests that (5.17) can be represented by a constrained variational principle.

Theorem 6.1. *Consider the constrained action principle*

$$\mathcal{A} = \iint \mathcal{L}' \,\mathrm{d}h\,\mathrm{d}t, \tag{6.2}$$

where the constrained Lagrangian $\mathcal{L}'$ has the form

$$\mathcal{L}' = \mathcal{L}_0 - \left(\sum_{i=1}^{4} \lambda_i E_i + \mu_i F_i\right). \tag{6.3}$$

Here, $\mathcal{L}_0$ is the Lagrangian density in (2.16) for an ideal gas with adiabatic index γ, namely

$$\mathcal{L}_0 = \frac{1}{2}u^2 - \frac{pq}{\gamma - 1}, \tag{6.4}$$

and the E_i and F_i are given by

$$\begin{aligned}
&E_1 = \tilde{u}^1_h - q, && F_1 = \tilde{u}^1_t - u,\\
&E_2 = \tilde{u}^2_h - \left(\frac{1}{2}u^2 + \frac{pq}{\gamma - 1}\right), && F_2 = \tilde{u}^2_t + pu,\\
&E_3 = \tilde{u}^3_h - u, && F_3 = \tilde{u}^3_t + p,\\
&E_4 = \tilde{u}^4_h + \tilde{u}^1 - tu, && F_4 = \tilde{u}^4_t + pt.
\end{aligned} \tag{6.5}$$

The constraint equations $E_i = 0$ and $F_i = 0$ correspond to the potential form of the mass, energy, momentum and center of mass conservation laws in the cover system (5.17). The gas pressure p is given by the equation of state:

$$p = N_1 q^{-\gamma} \exp[\bar{S}(h)] \tag{6.6}$$

($\bar{S} = S/C_v$ is a dimensionless form of the gas entropy). The constrained variational principle (6.2)–(6.6) is equivalent to the Sjöberg and Mahomed (2004) cover system (5.17) provided that the Lagrange multipliers $\{\mu_j, \lambda_j\}$ $(1 \leqslant j \leqslant 4)$ have the form

$$\begin{aligned}
&\mu_1 = -u + D_h(p\Omega_2) - \Omega_4, && \lambda_1 = -p - D_t(p\Omega_2),\\
&\mu_2 = D_h(\Omega_2), && \lambda_2 = -D_t(\Omega_2),\\
&\mu_3 = -D_h(t\Omega_4 + u\Omega_2), && \lambda_3 = D_t(t\Omega_4 + u\Omega_2),\\
&\mu_4 = D_h(\Omega_4), && \lambda_4 = -D_t(\Omega_4).
\end{aligned} \tag{6.7}$$

In (6.7), the functions Ω_2 and Ω_4 are either (a) differentiable functions of h and t or (b) functions of known potentials $\{\tilde{u}^j\}$ associated with conserved vectors $\mathbf{T}_j = \left(\tilde{u}^j_h, -\tilde{u}^j_t\right)$.

Proof. The stationary point conditions $\delta\mathcal{A}/\delta\mu_j = 0$ and $\delta\mathcal{A}/\delta\lambda_j = 0$ $(1 \leqslant j \leqslant 4)$ requires that the $\tilde{u}^j$ satisfy the potential equations (5.17), (i.e. $E_i = 0$, $F_i = 0$, $1 \leqslant i \leqslant 4$). The stationary point conditions $\delta\mathcal{A}/\delta u = 0$ and $\delta\mathcal{A}/\delta q = 0$ require

$$\frac{\delta\mathcal{A}}{\delta u} \equiv u + \mu_1 + \lambda_2 u - \mu_2 p + \lambda_3 + t\lambda_4 = 0, \tag{6.8}$$

$$\frac{\delta\mathcal{A}}{\delta q} \equiv p + \lambda_1 - \lambda_2 p + \frac{\gamma p u}{q}\mu_2 + \frac{\gamma p}{q}\mu_3 + \frac{\gamma p t}{q}\mu_4 = 0. \tag{6.9}$$

The stationary point conditions $\delta\mathcal{A}/\delta\tilde{u}^j = 0$ $(1 \leqslant j \leqslant 4)$ reduce to

$$\frac{\delta\mathcal{A}}{\delta\tilde{u}^1} \equiv D_t\mu_1 + D_h\lambda_1 - \lambda_4 = 0, \tag{6.10}$$

$$\frac{\delta\mathcal{A}}{\delta\tilde{u}^j} \equiv D_t\mu_j + D_h\lambda_j = 0, \qquad j = 2, 3, 4. \tag{6.11}$$

To derive (6.7) we first note from (6.11) that the Lagrange multipliers $\{\mu_j, \lambda_j\}$ $(j = 2, 3, 4)$ are conserved vectors, and hence can be written in terms of potentials Ω_j in the form

$$(\mu_j, \lambda_j) = (D_h\Omega_j, -D_t\Omega_j), \qquad j = 2, 3, 4. \tag{6.12}$$

Taking into account the form of (μ_4, λ_4) in (6.12), it follows that

$$\begin{aligned} &(\mu_1 + \Omega_4, \lambda_1) = (D_h\Omega_1, -D_t\Omega_1), \\ &\mu_1 = D_h\Omega_1 - \Omega_4, \qquad \lambda_1 = -D_t\Omega_1. \end{aligned} \tag{6.13}$$

Using (6.12)–(6.13) for $\{\mu_j, \lambda_j\}$, (6.8) reduces to the conservation law:

$$D_t(\tilde{u}^1 - t\Omega_4 - \Omega_3 - u\Omega_2) + D_h(\Omega_1 - p\Omega_2) = 0, \tag{6.14}$$

(note that $u = \tilde{u}^1_t$ and $\tilde{u}^1 \equiv x(h, t)$), and hence there exists a potential Λ such that

$$\begin{aligned} &\tilde{u}^1 - t\Omega_4 - \Omega_3 - u\Omega_2 = \Lambda_h, \qquad \Omega_1 - p\Omega_2 = -\Lambda_t. \\ &\Omega_1 = p\Omega_2 - \Lambda_t, \qquad \Omega_3 = \tilde{u}^1 - t\Omega_4 - u\Omega_2 - \Lambda_h. \end{aligned} \tag{6.15}$$

Using (6.12)–(6.13) and (6.15) in (6.9), we obtain the wave equation

$$\Lambda_{tt} - \frac{\gamma p}{q}\Lambda_{hh} + (\gamma + 1)p = 0, \tag{6.16}$$

for Λ. From (2.21)

$$w_{tt} + p = 0, \qquad w_{hh} = x_h = q, \qquad w_{tt} - \frac{\gamma p}{q}w_{hh} = -(\gamma + 1)p, \tag{6.17}$$

and hence we identify $\Lambda = w(h, t)$. Using (6.1) we obtain

$$\begin{aligned} &\Lambda_t = w_t = \tilde{u}^3, \qquad \Lambda_h = w_h = \tilde{u}^1. \\ &\Omega_1 = p\Omega_2 - \tilde{u}^3, \qquad \Omega_3 = -t\Omega_4 - u\Omega_2. \end{aligned} \tag{6.18}$$

Substituting formulas (6.18) for Ω_1 and Ω_3 in (6.12)–(6.13) gives formulas (6.7) for the Lagrange multipliers $\{\mu_j, \lambda_j\}$ $(1 \leqslant j \leqslant 4)$. This completes the proof. □

Comment 1

Although not obvious from the proof of theorem 6.1, the theorem implicitly assumes that not only are the constraint equations $E_i = 0$ and $F_i = 0$ satisfied but also the differential consequences of these equations are also satisfied (e.g. $D_t(E_i) = D_t(F_i) = 0$ and $D_h(E_i) = D_h(F_i) = 0$). This can be verified *a posteriori*, by using the solutions (6.7) for the Lagrangian constraint functions for the simple case for which $\Omega_2 = \mu_2 h - \lambda_2 t$ and $\Omega_4 = \mu_4 h - \lambda_4 t$, where (μ_2, λ_2) and (μ_4, λ_4) are constant vectors and re-evaluating the variational derivatives for $\delta\mathcal{A}/\delta u$ and $\delta\mathcal{A}/\delta q$ in (6.8) and (6.9).

6.1. Applications

As a first application of theorem 6.1, consider the simplest case in which $\Omega_2 = \Omega_4 = 0$. In this case,

$$\mu_1 = -u, \qquad \lambda_1 = -p, \qquad \mu_j = \lambda_j = 0, \qquad j = 2, 3, 4. \tag{6.19}$$

The constrained Lagrangian density $\mathcal{L}'$ in (6.3) reduces to

$$\mathcal{L}' = \frac{1}{2}u^2 - \frac{pq}{\gamma - 1} + p(\tilde{u}^1_h - q) + u(\tilde{u}^1_t - u), \tag{6.20}$$

where $p\ (q, h)$ is given by (6.4). The variational equations $\delta\mathcal{A}/\delta\tilde{u}^1 = 0$, $\delta\mathcal{A}/\delta q = 0$ and $\delta\mathcal{A}/\delta u = 0$ reduce to

$$\frac{\delta\mathcal{A}}{\delta\tilde{u}^1} = -(u_t + p_h) = 0, \tag{6.21}$$

$$\frac{\delta\mathcal{A}}{\delta q} = -\frac{\gamma p}{q}\left(\tilde{u}^1_h - q\right) = 0, \tag{6.22}$$

$$\frac{\delta\mathcal{A}}{\delta u} = \tilde{u}^1_t - u = 0. \tag{6.23}$$

For this form of the variational principle, we recover the original system of partial differential equations by requiring $\tilde{u}^1$ to have continuous second partial derivatives (note that we have replaced the entropy-related evolution equation for p_t in (5.17) by the equation of state (6.6)). Clearly, we require non-zero Ω_2 and Ω_4 to recover the complete Sjöberg and Mahomed (2004) cover system. Equations (6.21)–(6.23) can also be cast in Hamiltonian form by using a Legendre transformation.

One can show that the Galilean boost symmetry

$$\tilde{X}_4 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial\tilde{u}^1} + \tilde{u}^3\frac{\partial}{\partial\tilde{u}^2} + h\frac{\partial}{\partial\tilde{u}^3} \tag{6.24}$$

is a divergence symmetry of the action. The corresponding conservation law is the center of mass conservation law, which is associated with the potential $\tilde{u}^4$.

Noether's theorem and conservation laws

If the extended symmetry operator $\tilde{X}$ is a divergence symmetry of the action (6.2) then Noether's theorem implies the existence of a conservation law, with conserved vector (I_0, I_1) of the form

$$I_0 = W^t + V^t\mathcal{L}' + \Lambda^t, \qquad I_1 = W^h + V^h\mathcal{L}' + \Lambda^h, \tag{6.25}$$

where

$$\begin{aligned} W^t &= \hat{V}^{u^\alpha}\frac{\delta\mathcal{L}'}{\delta u^\alpha_t} + \hat{V}^{u^\alpha_{jt}}\frac{\delta\mathcal{L}'}{\delta u^\alpha_{jt}} + \dots, \\ W^h &= \hat{V}^{u^\alpha}\frac{\delta\mathcal{L}'}{\delta u^\alpha_h} + \hat{V}^{u^\alpha_{jh}}\frac{\delta\mathcal{L}'}{\delta u^\alpha_{jh}} + \dots, \end{aligned} \tag{6.26}$$

$$\frac{\delta\mathcal{L}'}{\delta\psi} = \frac{\partial\mathcal{L}'}{\partial\psi} - D_j\left(\frac{\partial\mathcal{L}'}{\partial\psi_j}\right) + D_jD_k\left(\frac{\partial\mathcal{L}'}{\partial\psi_{jk}}\right) + \dots \tag{6.27}$$

(see, e.g., Ibragimov (1994) and Bluman and Kumei (1989)). The condition for $\tilde{X}$ to be a divergence symmetry of the action may be written in the form:

$$\tilde{X}\mathcal{L}' + (D_tV^t + D_hV^h)\mathcal{L}' + D_t\Lambda^t + D_h\Lambda^h = 0, \tag{6.28}$$

where Λ^t and Λ^h are functions corresponding to an infinitesimal divergence transformation of the Lagrangian.

Theorem 6.2. *A divergence symmetry $\tilde{X}$ of the constrained action (6.2) satisfying the divergence symmetry condition (6.28) implies via Noether's theorem, the conservation law:*

$$D_t I_0 + D_h I_1 = 0, \tag{6.29}$$

where

$$I_0 = u\hat{V}^x + V^t\mathcal{L}_0 + \Lambda^t, \qquad I_1 = p\hat{V}^x + V^h\mathcal{L}_0 + \Lambda^h, \tag{6.30}$$

$$\hat{V}^x = V^x - (V^t D_t + V^h D_h)x(h,t), \qquad x(h,t) \equiv \tilde{u}^1. \tag{6.31}$$

Here $\hat{V}^x$ is the canonical symmetry generator for $x(h, t)$ taking into account the Lie extension formulas for the transformation of the potentials $\{\tilde{u}^j : 1 \leqslant j \leqslant 4\}$ for the Sjöberg and Mahomed (2004) cover system (5.17).

Proof. The symmetry operator $\tilde{X}$ is a symmetry of the cover system (5.17) and hence

$$\tilde{X}(E_i) = \tilde{X}(F_i) = 0, \qquad 1 \leqslant i \leqslant 4. \tag{6.32}$$

Using the fact that $E_i = F_i = 0$ on the solution manifold, and using (6.32) it follows that $\mathcal{L}'$ can be replaced by $\mathcal{L}_0$ in (6.28).

For the constrained variational principle (6.2) we find

$$W^t = \hat{V}^{\tilde{u}^k}\frac{\partial\mathcal{L}'}{\partial\tilde{u}^k_t} = -\mu_k\hat{V}^{\tilde{u}^k}, \qquad W^h = \hat{V}^{\tilde{u}^k}\frac{\partial\mathcal{L}'}{\partial\tilde{u}^k_h} = -\lambda_k\hat{V}^{\tilde{u}^k} \tag{6.33}$$

are the form of the surface vector terms in Noether's theorem, where the $\hat{V}^{\tilde{u}^k}$ are the canonical Lie symmetry generators for the $\tilde{u}^k$ $(1 \leqslant k \leqslant 4)$. In (6.33), we use the Einstein summation convention for repeated indices and $1 \leqslant k \leqslant 4$ in the sums over k. From (6.25) and (6.33), the conserved vector (I_0, I_1) has the form

$$\begin{aligned} I_0 &= -\mu_k\hat{V}^{\tilde{u}^k} + V^t\mathcal{L}_0 + \Lambda^t, \\ I_1 &= -\lambda_k\hat{V}^{\tilde{u}^k} + V^h\mathcal{L}_0 + \Lambda^h. \end{aligned} \tag{6.34}$$

Taking into account the form (6.7) and (6.18) of the Lagrange multipliers in terms of the potentials $\{\Omega_k\}$, the conserved vector (I_0, I_1) in (6.34) may be expressed in the form

$$I_0 = -D_h\left(\Omega_k\hat{V}^{\tilde{u}^k}\right) + \Phi, \qquad I_1 = D_t\left(\Omega_k\hat{V}^{\tilde{u}^k}\right) + \Psi, \tag{6.35}$$

where

$$\begin{aligned} \Phi &= \Omega_k D_h\left(\hat{V}^{\tilde{u}^k}\right) + \Omega_4\hat{V}^{\tilde{u}^1} + V^t\mathcal{L}_0 + \Lambda^t, \\ \Psi &= -\Omega_k D_t\left(\hat{V}^{\tilde{u}^k}\right) + V^h\mathcal{L}_0 + \Lambda^h. \end{aligned} \tag{6.36}$$

Using formulas (6.18) for Ω_1 and Ω_3 in (6.36), we obtain

$$\begin{aligned} \Phi &= c_2\Omega_2 + c_4\Omega_4 + \left[-\tilde{u}^3 D_h\left(\hat{V}^{\tilde{u}^1}\right) + V^t\mathcal{L}_0 + \Lambda^t\right], \\ \Psi &= d_2\Omega_2 + d_4\Omega_4 + \left[\tilde{u}^3 D_t\left(\hat{V}^{\tilde{u}^1}\right) + V^h\mathcal{L}_0 + \Lambda^h\right], \end{aligned} \tag{6.37}$$

where

$$\begin{aligned} c_2 &= p\hat{V}^{\tilde{u}^1_h} + \hat{V}^{\tilde{u}^2_h} - \hat{V}^{\tilde{u}^3_h}, \qquad & c_4 &= \hat{V}^{\tilde{u}^1} + \hat{V}^{\tilde{u}^4_h} - t\hat{V}^{\tilde{u}^3_h}, \\ d_2 &= -p\hat{V}^{\tilde{u}^1_t} - \hat{V}^{\tilde{u}^2_t} + u\hat{V}^{\tilde{u}^3_t}, \qquad & d_4 &= \hat{V}^{\tilde{u}^4_t} - t\hat{V}^{\tilde{u}^3_t}. \end{aligned} \tag{6.38}$$

In the derivation of (6.37)–(6.38), we used the Lie extension formulas: $\hat{V}^{\tilde{u}^\alpha_j} = D_j(\hat{V}^{\tilde{u}^\alpha})$. where $(x^1, x^2) = (t, h)$. Using the cover system (5.17) in (6.38), we find

$$c_2 = c_4 = d_2 = d_4 = 0, \tag{6.39}$$

(note for example that $\hat{V}^{pt} = t\hat{V}^p$ for canonical Lie transformations, as the independent variables do not change). Equations (6.39) arise from the Lie invariance of the cover system with respect to $\tilde{X}$. It now follows that

$$\begin{aligned} \Phi &= -D_h\left(\tilde{u}^3\hat{V}^{\tilde{u}^1}\right) + \hat{\Phi}, \qquad & \hat{\Phi} &= u\hat{V}^{\tilde{u}^1} + V^t\mathcal{L}_0 + \Lambda^t, \\ \Psi &= D_t\left(\tilde{u}^3\hat{V}^{\tilde{u}^1}\right) + \hat{\Psi}, \qquad & \hat{\Psi} &= p\hat{V}^{\tilde{u}^1} + V^h\mathcal{L}_0 + \Lambda^h. \end{aligned} \tag{6.40}$$

Both (Φ, Ψ) and $(\hat{\Phi}, \hat{\Psi})$ are conserved vectors. Re-naming $(\hat{\Phi}, \hat{\Psi}) = (I_0, I_1)$ and noting $\tilde{u}^1 \equiv x$ we obtain the conserved vector (6.30). This completes the proof. □

Example. As an application of the above results on Noether's theorem, consider the symmetry operator $\tilde{X}_{11}$ in (5.18), which is a symmetry of the cover system (5.17). Evaluating $\tilde{X}_{11}(\mathcal{L}_0)$, we obtain

$$\tilde{X}_{11}(\mathcal{L}_0) + (D_tV^t + D_hV^h)\mathcal{L}_0 + D_h\left[\tfrac{1}{2}(\tilde{u}^3)^2\right] = 0, \tag{6.41}$$

which establishes that $\tilde{X}_{11}$ is a divergence symmetry of the action (see discussion in (6.32) *et seq.*), for which

$$\Lambda^t = 0, \qquad \Lambda^h = \tfrac{1}{2}(\tilde{u}^3)^2. \tag{6.42}$$

For the symmetry operator $\tilde{X}_{11}$

$$V^t = 0, \qquad V^h = -h^2, \qquad \hat{V}^{\tilde{u}^1} = \tilde{u}^4 - t\tilde{u}^3 + h\tilde{u}^1 + h^2q. \tag{6.43}$$

The conserved vector (I_0, I_1) by theorem 6.2 is

$$\begin{aligned} I_0 &= u(\tilde{u}^4 - t\tilde{u}^3 + h\tilde{u}^1 + h^2q), \\ I_1 &= p(\tilde{u}^4 - t\tilde{u}^3 + h\tilde{u}^1 + h^2q) - h^2\left(\frac{1}{2}u^2 - \frac{pq}{\gamma - 1}\right) + \frac{1}{2}(\tilde{u}^3)^2. \end{aligned} \tag{6.44}$$

It is straightforward to verify that (I_0, I_1) satisfies the conservation law (6.29).

Comment

The constrained variational principle (6.2) expresses the Lagrange multipliers in terms of two arbitrary potentials Ω_2 and Ω_4. However, the potentials Ω_2 and Ω_4 do not appear in the corresponding Noether theorem 6.2, because of the Lie invariance conditions for the cover system. Thus, Ω_2 and Ω_4 are analogous to gauge potentials.

Example. Denote the conserved vector $\mathbf{T}^*_\alpha(\tilde{X}_\beta)$ as the new conserved vector obtained by applying $\tilde{X}_\beta$ to the known conserved vector $\mathbf{T}_\alpha$ using formula (5.3). Then by applying the Galilean boost symmetry $\tilde{X}_4$ of (B.1) to the conserved vector $\mathbf{T}_{11} = (I_0, I_1)$ in (6.44), we obtain the conserved vector:

$$\mathbf{T}^*_{11}(\tilde{X}_4) = (\tilde{u}^4 - t\tilde{u}^3 + h\tilde{u}^1 + h^2q, h\tilde{u}^3 - uh^2) = (\psi_h, -\psi_t), \tag{6.45}$$

where

$$\psi = h(\tilde{u}^4 - t\tilde{u}^3 + h\tilde{u}^1). \tag{6.46}$$

Comment

The potential ψ can be determined by using the homotopy formula:

$$\psi = \int_0^1 \frac{\mathrm{d}\lambda}{\lambda}(h'T'^1 - t'T'^2)|_{(t',h')=(\lambda t,\lambda h)}, \tag{6.47}$$

(e.g. Olver (1993), equation (1.69)) and by noting that

$$t' \frac{\partial}{\partial t'} + h' \frac{\partial}{\partial h'} = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}. \tag{6.48}$$

Example. Applying the scaling symmetry operator $\tilde{X}_{(s)}$ to $\mathbf{T}_{11} = (I_0, I_1)$ gives the new conserved vector:

$$\mathbf{T}^*_{11}(\tilde{X}_{(s)}) = (3\alpha_3 + 2\alpha_6 + 3\alpha_3)\mathbf{T}_{11}. \tag{6.49}$$

Note that $\mathbf{T}^*_{11}(\tilde{X}_{(s)})$ is trivial, because it is a multiple of the known conserved vector $\mathbf{T}_{11}$. Also note that

$$[\tilde{X}_{(s)}, \tilde{X}_{11}] = (\alpha_3 + 2\alpha_5 + \alpha_6)\tilde{X}_{11}. \tag{6.50}$$

The symmetry operator $\tilde{X}_{11}$ is associated with $\mathbf{T}_{11}$ in the sense that $\mathbf{T}^*_{11}(\tilde{X}_{11}) = 0$, and the the commutator result (6.50) implies $\mathbf{T}^*_{11}(\tilde{X}_{(s)})$ is trivial by theorem 4 of Kara and Mahomed (2002).

7. Concluding remarks

Scaling symmetries of the one-dimensional ideal gas dynamics equations for a gas with adiabatic index γ were used to obtain Lagrangian and Eulerian conservation laws. Conditions for a linear combination of the three scaling symmetries to be a variational or divergence symmetry of the action were established. In these cases, the conservation laws were derived using Noether's theorem. The Eulerian conservation laws can be determined once the Lagrangian form of the conservation law has been obtained (see proposition (4.2) and Padhye and Morrison (1996a, 1996b)). The scaling symmetry conservation laws only apply for special initial entropy distribution for the gas. For the case $\gamma = 3$, there is no constraint on the initial entropy distribution.

In section 5, we used the methods developed by Kara and Mahomed (2002) and Sjöberg and Mahomed (2004) to derive a nonlocal conservation law (5.23) obtained by applying the nonlocal symmetry operator X_{11} of Sjöberg and Mahomed (2004) to the scaling symmetry conserved vector. The nonlocal symmetry X_{11} only applies for a specific dependence of the entropy $S(h)$ on the Lagrangian mass coordinate h (see (5.21) *et seq.*). The conserved scaling symmetry vector (4.22) could be used to extend the cover system (5.17) of Sjöberg and Mahomed (2004) to search for further symmetries of the expanded cover system. The conservation laws obtained in the present paper presumably could be obtained by other methods (e.g. the direct method of Anco and Bluman (2002)).

In section 6, we developed a constrained variational principle, incorporating the mass, energy, momentum and center of mass conservation laws by means of Lagrange multipliers, which provides a variational formulation of the Sjöberg and Mahomed (2004) cover system (5.17). The requirements that the action be stationary with respect to the potentials $\tilde{u}^k$, $(1 \leqslant k \leqslant 4)$ and with respect to u and q delimited the allowed functional forms for the Lagrange multipliers. Noether's theorem for the constrained action principle has the same form as the classical Noether theorem, except that the Lie symmetry operators now incorporate the effects of the nonlocal potentials. A nonlocal conservation law for the symmetry X_{11} was obtained by Noether's theorem.

In appendix , we show that the conditions for similarity solutions of the nonlinear wave equation (3.13) (which is equivalent to the Euler equation $x_{tt} + p_h = 0$) to possess a first integral is different than the condition (4.17) for the equations to possess a conservation law.

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Appendix A

In this appendix we derive (5.3) using differential forms (see e.g. Harrison and Estabrook (1971) and Kara and Mahomed (2000)). Conservation laws of the form (5.2) are associated with the vanishing of the exterior derivative of the $n-1$ form:

$$\Omega = T \lrcorner\, \omega, \qquad \omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \ldots \wedge \mathrm{d}x^n, \tag{A.1}$$

where $T = T^i D_i$ is a contravariant vector field and ω is an n-form. Taking the exterior derivative of (A.1) gives

$$\mathrm{d}\Omega = \nabla\cdot T\ \omega. \tag{A.2}$$

The $n-1$ form Ω is closed if $\mathrm{d}\Omega = 0$, which in turn implies the conservation law $\nabla\cdot T = 0$. To derive (5.3), consider the condition for the $(n-1)$ form Ω to be invariant under the Lie symmetry operator X meaning that the Lie derivative of Ω with respect to vector field X is zero, i.e. $\mathcal{L}_X(\Omega) = 0$. Noting that

$$\begin{aligned}\mathcal{L}_X(\Omega) &= \mathcal{L}_X(T \lrcorner\, \omega) = \mathcal{L}_X(T) \lrcorner\, \omega + T \lrcorner \mathcal{L}_X(\omega)\\ &= [X, T] \lrcorner\, \omega + T \nabla\cdot\xi \lrcorner\, \omega\\ &= \left[X(T^i) - T^j D_j(\xi^i) + T^i \nabla\cdot\xi\right] D_i \lrcorner\, \omega = T^* \lrcorner\, \omega,\end{aligned} \tag{A.3}$$

we identify the vector field $T^* = T^{*i} D_i$ in (A.3) where

$$T^{*i} = [X(T^i) - T^j D_j(\xi^i) + T^i \nabla\cdot\xi], \tag{A.4}$$

which is formula (5.3) for T^{*i}. From (A.3) we find that $\mathcal{L}_X(\Omega) = 0$ if $T^{*i} = 0$, in which case we identify the symmetry X with the conservation law $\nabla\cdot T = 0$. More generally, $\mathcal{L}_X(\Omega) \neq 0$. Next using the formula $\mathcal{L}_X(\mathrm{d}\Omega) = d(\mathcal{L}_X\Omega)$ and using (A.2) for $\mathrm{d}\Omega$, we obtain

$$\mathcal{L}_X[\nabla\cdot T\omega] = d(T^* \lrcorner\, \omega) = \mathrm{d}\Omega^* = (D_i T^{*i})\omega. \tag{A.5}$$

From (A.5), it follows that if T is a conserved vector, so is T^* a conserved vector. This proves the result alluded to in (5.3) *et seq*.

Appendix B

In this appendix, we list the Lie extension formulas for the symmetry operators $\{X_i : 1 \leqslant i \leqslant 6,\ i = 11\}$. Using the result (5.5) derived by Sjöberg and Mahomed (2004), we obtain

$$\begin{aligned}\tilde{X}_1 &= \frac{\partial}{\partial t} + \tilde{u}^3 \frac{\partial}{\partial \tilde{u}^4},\\ \tilde{X}_2 &= \frac{\partial}{\partial \tilde{u}^1} - h \frac{\partial}{\partial \tilde{u}^4},\\ \tilde{X}_3 &= t\frac{\partial}{\partial t} + h\frac{\partial}{\partial h} + \tilde{u}^1 \frac{\partial}{\partial \tilde{u}^1} + \tilde{u}^2 \frac{\partial}{\partial \tilde{u}^2} + \tilde{u}^3 \frac{\partial}{\partial \tilde{u}^3} + 2\tilde{u}^4 \frac{\partial}{\partial \tilde{u}^4},\end{aligned}$$

$$\tilde{X}_4 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial \tilde{u}^1} + \tilde{u}^3\frac{\partial}{\partial \tilde{u}^2} + h\frac{\partial}{\partial \tilde{u}^3},$$
$$\tilde{X}_5 = t\frac{\partial}{\partial t} + 2h\frac{\partial}{\partial h} - u\frac{\partial}{\partial u} - 2q\frac{\partial}{\partial q} + \tilde{u}^3\frac{\partial}{\partial \tilde{u}^3} + 2\tilde{u}^4\frac{\partial}{\partial \tilde{u}^4},$$
$$\tilde{X}_6 = h\frac{\partial}{\partial h} + p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} + \tilde{u}^2\frac{\partial}{\partial \tilde{u}^2} + \tilde{u}^3\frac{\partial}{\partial \tilde{u}^3} + \tilde{u}^4\frac{\partial}{\partial \tilde{u}^4},$$
$$\tilde{X}_{11} = -h^2\frac{\partial}{\partial h} - hp\frac{\partial}{\partial p} + (hu - \tilde{u}^3)\frac{\partial}{\partial u} + 3hq\frac{\partial}{\partial q}$$
$$+ (\tilde{u}^4 - \tilde{u}^3 t + \tilde{u}^1 h)\frac{\partial}{\partial \tilde{u}^1} - \frac{1}{2}(\tilde{u}^3)^2\frac{\partial}{\partial \tilde{u}^2} - \tilde{u}^3 h\frac{\partial}{\partial \tilde{u}^3} - \tilde{u}^4 h\frac{\partial}{\partial \tilde{u}^4}. \tag{B.1}$$

The symmetry operator

$$\tilde{X}_{(s)} = \alpha_3\tilde{X}_3 + \alpha_5\tilde{X}_5 + \alpha_6\tilde{X}_6 \tag{B.2}$$

can be written down using (B.1). The symmetry operators $\{X_i : 1 \leqslant i \leqslant 6\}$ were obtained by transforming the Eulerian symmetry operators (2.5)–(2.7) using the Lagrangian map (see section 3 and Webb and Zank (2007)). For the conservation law cases $\alpha_5 + \alpha_6 + 2\alpha_3 = 0$ and the equation of state depends on the parameters $\{\alpha_3, \alpha_5, \alpha_6\}$.

The non-zero commutators in (B.1) are

$$\begin{aligned} &[\tilde{X}_3, \tilde{X}_{11}] = \tilde{X}_{11}, \qquad [\tilde{X}_5, \tilde{X}_{11}] = 2\tilde{X}_{11}, \qquad [\tilde{X}_6, \tilde{X}_{11}] = \tilde{X}_{11},\\ &[\tilde{X}_1, \tilde{X}_3] = \tilde{X}_1, \qquad [\tilde{X}_1, \tilde{X}_4] = \tilde{X}_2, \qquad [\tilde{X}_1, \tilde{X}_5] = \tilde{X}_1,\\ &[\tilde{X}_2, \tilde{X}_3] = \tilde{X}_2, \qquad [\tilde{X}_4, \tilde{X}_5] = -\tilde{X}_4. \end{aligned} \tag{B.3}$$

The formula

$$[\tilde{X}_{(s)}, \tilde{X}_{11}] = (\alpha_3 + 2\alpha_5 + \alpha_6)\tilde{X}_{11} \tag{B.4}$$

gives the commutator for $\tilde{X}_{(s)}$ and $\tilde{X}_{11}$. The operators used by Sjöberg and Mahomed (2004) denoted by $\{\tilde{X}_i^S\}$ are given by

$$\begin{aligned} &\tilde{X}_1^S = \tilde{X}_1, \qquad \tilde{X}_2^S = \tilde{X}_3, \qquad \tilde{X}_3^S = \tilde{X}_2, \qquad \tilde{X}_4^S = \tilde{X}_4,\\ &\tilde{X}_5^S = \tilde{X}_7 \equiv \frac{\partial}{\partial h}, \qquad \tilde{X}_6^S = 2\tilde{X}_3 - \tilde{X}_5, \qquad \tilde{X}_7^S = \tilde{X}_6. \end{aligned} \tag{B.5}$$

We have not included the fluid relabeling symmetry $X_7 = \partial_h$ in (B.1), which applies if the entropy of the gas $S = \text{const}$. If $S = \text{const}$., one must add the symmetries $\{\partial_{\tilde{u}^j} : j = 2, 3, 4\}$ to the list of symmetry generators. It is straightforward to determine the commutators in this case also. A larger symmetry algebra occurs for the cases $\gamma = 3$ and $\gamma = -1$ (e.g. Sjöberg and Mahomed (2004)). Note that the symmetry algebra depends on the equation of state.

Appendix C

Consider the classical similarity solutions of the nonlinear wave equation (3.13) for the scaling symmetries. This equation is the Lagrangian momentum equation $x_{tt} + p_h = 0$. We show that condition (4.14) for the equation to possess a conservation law is not the same as the condition for the similarity reduced ordinary differential equation to possess a first integral.

The general similarity solution of (3.13) is obtained by integrating the group characteristics:

$$\frac{\mathrm{d}t}{V^t} = \frac{\mathrm{d}h}{V^h} = \frac{\mathrm{d}x}{V^x}, \tag{C.1}$$

where V^t, V^h and V^x are the group generators (3.6) (e.g. Bluman and Kumei (1989)). Integration of the equation $\mathrm{d}h/\mathrm{d}t = V^h/V^t$ gives the similarity variable

$$\eta = (h + d_1)t^{-\nu_2}, \qquad \nu_2 = \frac{\delta_2}{\alpha}, \tag{C.2}$$

where η is constant on the characteristics. Similarly, integrating the equation $dx/dt = V^x/V^t$ it follows that the similarity solution has the form

$$x(h,t) = t^{\nu_3}F(\eta), \qquad \nu_3 = \frac{\alpha_3}{\alpha}, \tag{C.3}$$

where $F(\eta)$ is an arbitrary function of η, which is chosen so that $x(h,t)$ satisfies (3.13).

Substitution of the solution ansatz (C.2)–(C.3) for $x(h,t)$ into the nonlinear wave equation (3.13) results in the ordinary differential equation:

$$\nu_2^2\eta^2F'' + \nu_2(\nu_2 + 1 - 2\nu_3)\eta F' - N_1\eta^{-2\nu-2}(F')^{-\gamma-1}(\gamma\eta^2F'' + 2\nu\eta F') = 0, \tag{C.4}$$

for $F(\eta)$, where $F' = \mathrm{d}F/\mathrm{d}\eta$ and $F'' = \mathrm{d}^2F/\mathrm{d}\eta^2$. Equation (C.4) can also be written in the more suggestive form:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\nu_2^2\eta^2\frac{\mathrm{d}F}{\mathrm{d}\eta} + \eta F[\nu_3^2 - \nu_3 + (\nu_2+\nu_3)(\nu_2+\nu_3-1)] + N_1\eta^{-2\nu}\left[\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]^{-\gamma}\right) + (\nu_2+\nu_3)(\nu_2+\nu_3-1)F = 0. \tag{C.5}$$

Thus, if

$$(\nu_2+\nu_3)(\nu_2+\nu_3-1) = 0, \tag{C.6}$$

equation (C.4) possesses the first integral:

$$\nu_2^2\eta^2\frac{\mathrm{d}F}{\mathrm{d}\eta} + \eta F(\nu_3^2 - \nu_3) + N_1\eta^{-2\nu}\left[\frac{\mathrm{d}F}{\mathrm{d}\eta}\right]^{-\gamma} = \text{const}. \tag{C.7}$$

Since

$$\nu_2+\nu_3 = \frac{2\alpha_3 + 2\alpha_5 + \alpha_6}{\alpha}, \qquad \nu_2+\nu_3-1 = \frac{\alpha_3+\alpha_5+\alpha_6}{\alpha}, \tag{C.8}$$

it is clear that the conditions (C.6) are not equivalent to the condition (4.14): $\alpha_5+\alpha_6+2\alpha_3 = 0$ for a conservation law.

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